

The Birman-Murakami-Wenzl Algebra of Coxeter Type D

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Abstract

The paper defines a generic Birman-Wenzl algebra of Coxeter Type D and investigates its structure as a semi-simple algebra.

Definition 1 *The braid group ZD_n of Coxeter D-type has generators X_0, \dots, X_{n-1} with relations*

$$X_i X_j X_i = X_j X_i X_j \quad i, j \geq 1, |i - j| = 1 \quad (1)$$

$$X_0 X_2 X_0 = X_2 X_0 X_2 \quad (2)$$

$$X_0 X_j = X_j X_0 \quad j \neq 2 \quad (3)$$

$$X_i X_j = X_j X_i \quad |i - j| > 1, i, j \geq 1 \quad (4)$$

In addition, the Coxeter Group WD_n generators satisfy the quadratic relation for all $i \geq 0$

$$X_i^2 = 1 \quad (5)$$

Definition 2 *The braid group ZB_n of Coxeter B-type has generators Y, X_1, \dots, X_{n-1} and relations*

$$Y X_1 Y X_1 = X_1 Y X_1 Y \quad (6)$$

$$X_i X_j X_i = X_j X_i X_j \quad |i - j| = 1 \quad (7)$$

$$Y X_i = X_i Y \quad i \geq 2 \quad (8)$$

$$X_i X_j = X_j X_i \quad |i - j| > 1 \quad (9)$$

In addition, the Coxeter Group WB_n includes quadratic relations:

$$X_i^2 = 1 \quad (10)$$

$$Y^2 = 1 \quad (11)$$

Lemma 1 *There is an injective morphism of Coxeter groups $i : \text{WD}_n \rightarrow \text{WB}_n$ which maps $X_0 \mapsto YX_1Y, X_i \mapsto X_i$.*

Proof: The proof that i is a morphism is simple. We only check (??):

$$i(X_0X_1) = YX_1YX_1 = X_1YX_1Y = i(X_1)i(X_0) = i(X_1X_0)$$

and (2)

$$\begin{aligned} i(X_0X_2X_0) &= YX_1YX_2YX_1Y = YX_1X_2Y^2X_1Y = YX_1X_2X_1Y \\ &= YX_2X_1X_2Y = X_2YX_1YX_2 = i(X_2X_0X_2) \end{aligned}$$

As all relations preserve the parity of the number of Y in the words, it is clear that none of the words with an odd number of Y are in the image. On the other hand, words in WB_n with an even number of Y lie in the image: Chose a word in WB_n that has an even number of Y . Then we can choose the first and second occurrence of Y and using (8) we can bring them together until only a power of X_1 is in between so that we have the sub-word $w = YX_1^kY$. If k is even, then $w = 1$, else $w = YX_1Y = i(X_0)$. Hence, the subgroup of WB_n consisting of words with an even number of Y is isomorphic to WD_n . \square

Definition 3 *Let q denote an invertible element in an integral domain R . The D -type Hecke algebra HD_n over this ring is the quotient of the group algebra of ZD_n by the relation*

$$X_i^2 = (q-1)X_i + q, i \geq 0 \quad (12)$$

Definition 4 *Let q, p_0, p_1 denote invertible elements in an integral domain R . The B -type Hecke algebra HB_n over this ring is the quotient of the group algebra of ZB_n by the relations*

$$X_i^2 = (q-1)X_i + q \quad (13)$$

$$(Y - p_0)(Y - p_1) = 0 \quad (14)$$

Note that (14) is equivalent to $Y^2 = (Q-1)Y + Q$ with $p_0 = Q, p_1 = -1$ and this is a more common way of giving the quadratic relation.

We would like to extend the morphism from lemma 1 to the Hecke algebras.

Lemma 2 *The map $X_0 \mapsto kYX_1Y, X_i \mapsto X_i$ for some $k \in R$ defines an injective morphism $i : \text{HD}_n \rightarrow \text{HB}_n$ if $p_0 + p_1 = 0$ and $k = \frac{-1}{p_0p_1}$.*

Proof: Under the assumption on the parameters stated in the lemma the quadratic relation for Y simplifies to $Y^2 = -p_0p_1 = 1/k$. We have to check that $i(X_0) = kYX_1Y$ satisfies the correct quadratic relation:

$(kYX_1Y)^2 = k^2YX_1Y^2X_1Y = kYX_1^2Y = (q-1)kYX_1Y + qkY^2 = (q-1)kYX_1Y + q$. Bases of Hecke algebras may be labeled by words in the Coxeter group. Now, i is an injection in the group case and, due to the relation $Y^2 = 1/k$, the image of i closes as a subalgebra of HD_n . It consists of those words that have an even number of Y . \square

By the results of Ariki and Koike [1],[2] the algebra HB_n is semi-simple in the generic case. The simple HB_n Modules are parametrized by the set of pairs $T = (T_0, T_1)$ of Young diagrams of total size n . For such a pair T the simple module V_T has a basis of pairs of Young tableaux $t = (t_0, t_1)$ of shape T . The generator Y acts as multiplication with its eigenvalues (i.e. if $(Y - p_0)(Y - p_1) = 0$, then as multiplication with p_0 resp. p_1) depending on the tableaux containing 1, i.e. if 1 is contained in t_i it acts as multiplication with p_i . X_i acts as multiplication with q if i and $i+1$ are contained in the same row of the same tableaux, and as -1 if they are contained in the same column of the same tableaux. Otherwise the result of X_i acting on t is a linear combination of t and t' , where in t' the numbers $i, i+1$ are interchanged.

Using the morphism i we can view V_T as a module of HD_n given a restriction on the eigenvalues of Y .

Lemma 3 *Suppose that the eigenvalues of $Y \in \text{HD}_n$ satisfy $p_0^2 = p_1^2$, i.e. that $Y^2 = k = p_0^2 \in R$. Then the HD_n modules correspond to pairs of Young diagrams. The modules corresponding to $T = (T_0, T_1)$ and $T' = (T_1, T_0)$ are isomorphic. Furthermore, if $T_0 = T_1$ and 2 is a unit in the ground ring then the module V_T is HD_n -reducible $V_T = V_T^+ \oplus V_T^-$. The modules V_T^+ and V_T^- have the same dimension, but they are not equivalent.*

Proof: Consider the map $P : V_T \rightarrow V_{T'}, P(t_0, t_1) = (t_1, t_0)$. The definition of the action of $X_i, i \geq 1$ does obviously commute with P . Furthermore, $i(X_0) = YX_1Y$ acts as $p_0^2X_1$ or $p_1^2X_1$ if 1, 2 are in the same tableaux and as $p_0p_1X_1$ if they are in different tableaux. Thus assuming $p_0^2 = p_1^2$ we have that P and $i(X_0)$ commute. This proves the first claim. Suppose now that $T_0 = T_1$. We see that $P_{\pm} := (1 \pm P)/2$ are projectors on submodules V_T^{\pm} . Setting $q = 1$ one easily sees that these modules are not isomorphic. \square

For the rest of the paper we consider the algebra over the field of fraction of the ground ring.

Since i is an injection and the V_T form a complete set of simple module of HB_n , we conclude that all simple modules must be obtained in this way. In fact we have:

Proposition 4 *The set of pairwise non-equivalent absolutely irreducible HD_n representations is indexed by a set $I = I_0 \cup I_1$, where $I_0 := \{(T_0, T_1) \mid T_0 \neq T_1, |T_0| + |T_1| = n\}$ is the set of sets of two*

Young diagrams with total size n and $I_1 := \{T^s \mid s \in \{\pm 1\}, 2|T| = n\}$ is the set of \mathbb{Z}_2 labeled Young diagrams with exactly $n/2$ boxes.

Proof: I indexes the two types of modules that appeared in the proof of the preceding lemma. We show that they are irreducible. This is done by induction in the same fashion as [1][Proof of 3.9]: For small n the claim may be shown by direct calculations. Generators X_0, \dots, X_{n-2} generate a sub-algebra HD'_{n-1} of HD_n , which is a surjective image of HD_{n-1} . We divide the set of Young tableaux of shape T in two disjoint sets $T^{(l)}, T^{(r)}$ according to the position of n . They split V_T into a direct sum of two HD'_{n-1} -submodules and by induction assumption these are non-equivalent absolutely irreducible representations. Now assume that $W \subset V_T$ is a HD_n submodule. Viewed as a HD'_{n-1} module it must contain at least one of the modules generated by $T^{(l)}$ or $T^{(r)}$. The action of X_{n-1} , however, does not leave such a submodule stable but allows to generate any other Young tableaux from it. Hence W must be equal to V_T . The modules are pairwise inequivalent because their restrictions to HD'_{n-1} are by induction assumption inequivalent. \square

Note that I_0 consists of half of the HB_n modules labeled by two distinct diagrams. Each of the other HB_n modules gives rise to two modules of half the dimension. Hence, consistently, we arrive at the fact that the dimension of HD_n is half the dimension of HB_n , namely $2^{n-1}n!$.

Definition 5 Let R denote an integral domain with units $x, q, \lambda \in R$ such that with $\delta := q - q^{-1}$ the relation $(1 - x)\delta = \lambda - \lambda^{-1}$ holds. The D -type BMW algebra BD_n over this ring is the quotient of the R -algebra generated by $X_0^\pm, \dots, X_{n-1}^\pm, e_0, \dots, e_{n-1}$ subject to the following relations:

$$X_i X_j X_i = X_j X_i X_j \quad i, j \geq 1, |i - j| = 1 \quad (15)$$

$$X_0 X_2 X_0 = X_2 X_0 X_2 \quad (16)$$

$$X_0 X_1 = X_1 X_0 \quad (17)$$

$$e_0 X_1 = X_1 e_0 = e_0 \quad (18)$$

$$e_0 X_0 = X_0 e_1 = e_1 \quad (19)$$

$$X_i X_j = X_j X_i \quad |i - j| > 1, i, j \geq 1 \quad (20)$$

$$X_i e_i = e_i X_i = \lambda e_i \quad 0 \leq i \leq n - 1 \quad (21)$$

$$e_i X_j^{\pm 1} e_i = \lambda^{\mp 1} e_i \quad |i - j| = 1 \quad (22)$$

$$e_0 X_2^{\pm 1} e_0 = \lambda^{\mp 1} e_0 \quad (23)$$

$$e_2 X_0^{\pm 1} e_2 = \lambda^{\mp 1} e_2 \quad (24)$$

$$e_i^2 = x e_i \quad 0 \leq i \leq n - 1 \quad (25)$$

$$X_i^{-1} = X_i - \delta + \delta e_i \quad 0 \leq i \leq n - 1 \quad (26)$$

$$e_i e_j = e_j e_i \quad |i - j| > 1 \quad (27)$$

$$e_0 e_1 = e_1 e_0 \quad (28)$$

$$e_i X_j X_i = X_j^\pm X_i^\pm e_j \quad |i - j| = 1 \quad (29)$$

$$e_0 X_2 X_0 = X_2^\pm X_0^\pm e_2 \quad (30)$$

$$e_2 X_0 X_2 = X_0^\pm X_2^\pm e_0 \quad (31)$$

$$e_i e_j e_i = e_i \quad |i - j| = 1 \quad (32)$$

$$e_0 e_2 e_0 = e_0 \quad (33)$$

$$e_2 e_0 e_2 = e_2 \quad (34)$$

Most relations are directly transferred from other BMW algebras. And many of them are necessary to render the algebra finite dimensional. An exception is the pair (18),(18). Even with out them, the algebra is finite dimensional and the structure of the Coxeter graph simply suggests that they should commute (i.e. the first equal sign). However, omitting them would break down the nice relation to the corresponding B-type algebra. The algebra without these relations may, however, be an interesting object for further studies.

Lemma 5 *The quotient of BD_n by the ideal generated by e_0, e_1 is isomorphic to HD_n .*

Proof: The ideal contains all words that contain e_1 and thus also $e_2 = e_2 e_1 e_2$ and so forth. \square

The symmetry in the relations shows easily that there is an anti-involution $*$ that fixes all generators.

Next, we investigate the relation to the B-type Birman-Murakami-Wenzl algebra BB_n studied in [3]. We assume that we take a common ground ring R for both algebras. For the ground ring of BB_n it is required to have parameters $q, q_0, q_1, \lambda, x, \delta, A$ with relations $\delta = q - 1/q, x\delta = \delta - \lambda + 1/\lambda, A \cdot (1 - q_0\lambda) = q_1 x$. For details see [3]. We specialize parameters even further and put a prime on the algebra name to remind about this:

Lemma 6 *There is an morphism of algebras $i : \text{BD}_n \rightarrow \text{BB}'_n$ to the Birman-Murakami-Wenzl algebra of Coxeter type B with parameters chosen such that $q_0 = 1/\lambda, q_1 = 0$ given by $X_0 \mapsto \lambda Y X_1 Y, e_0 \mapsto \lambda Y e_1 Y, X_i \mapsto X_i, e_i \mapsto e_i, i > 0$.*

Proof: All calculations are straightforward. We give some examples:

- $i(X_0 X_2 X_0) = \lambda^2 Y X_1 Y X_2 Y X_1 Y = \lambda^2 Y X_1 X_2 Y^2 X_1 Y = \lambda Y X_1 X_2 X_1 Y = \lambda Y X_2 X_1 X_2 Y = \lambda X_2 Y X_1 Y X_2 = i(X_2 X_0 X_2)$
- $i(X_0 X_1) = \lambda Y X_1 Y X_1 = \lambda X_1 Y X_1 Y = i(X_1 X_0)$
- $i(e_1 X_0) = \lambda e_1 Y X_1 Y = e_1 = i(e_1)$

- $i(e_0 X_1) = \lambda Y e_1 Y X_1 = Y e_1 Y^{-1} = \lambda Y e_1 Y = i(e_0)$
- $i(X_0 e_0) = \lambda^2 Y X_1 Y Y e_1 Y = \lambda Y X_1 e_1 Y = \lambda^2 Y e_1 Y = \lambda i(e_0)$
- The relation $X_i^{-1} = X_i - \delta + \delta e_i$ is tested in its equivalent form $X_i^2 = 1 + \delta X_i - \delta \lambda e_i$:
 $i(X_0^2) = \lambda^2 Y X_1 Y^2 X_1 Y = \lambda Y X_1^2 Y = \lambda Y (1 + \delta X_i - \delta \lambda e_i) Y = 1 + \delta i(X_0) - \delta \lambda i(e_0)$
- $i(e_0 X_2 e_0) = \lambda^2 Y e_1 Y X_2 Y e_1 Y = \lambda Y e_1 X_2 e_1 Y = Y e_1 Y = \lambda^{-1} i(e_0)$
- $i(e_0^2) = \lambda^2 Y e_1 Y Y e_1 Y = \lambda Y e_1 e_1 Y = x \lambda Y e_1 Y = x \cdot i(e_0)$
- $0 = i(1 - X_0^2 + \delta X_0 - \delta e_0 X_0) = 1 - \lambda^2 Y X_1 Y Y X_1 Y + \delta \lambda Y X_1 Y - \delta \lambda^2 Y e_0 Y Y X_0 Y = 1 - \lambda Y X_1 X_1 Y + \delta \lambda Y X_1 Y - \delta \lambda Y e_1 X_1 Y = 1 - \lambda Y (1 + \delta X_1 - \delta \lambda e_1) Y + \delta \lambda Y X_1 Y - \delta \lambda^2 Y e_1 Y = 1 - \lambda Y^2 - \lambda \delta Y X_1 Y + \delta \lambda^2 Y e_1 Y + \delta \lambda Y X_1 Y - \delta \lambda^2 Y e_1 Y = 0$
- $i(e_0 X_1 e_0) = Y e_1 Y X_1 Y e_1 Y = Y e_1 X_1^{-1} X_1 Y X_1 Y e_1 Y = \lambda^{-1} Y e_1 X_1 Y X - 1 Y e_1 Y$
 $i([e_0, e_1]) = 0$ follows from $i([X_0, X_1]) = 0$ and (17).
- $i(e_0 X_2 X_0) = \lambda^2 Y e_1 Y X_2 Y X_1 Y = \lambda^2 Y e_1 Y^2 X_2 X_1 Y = \lambda Y e_1 X_2 X_1 Y = \lambda X_2 Y X_1 Y e_2 = i(X_2 X_0 e_2)$
- $i(e_0 e_2 e_0) = \lambda^2 Y e_1 Y e_2 Y e_1 Y = \lambda^2 Y e_1 Y Y e_2 e_1 Y = \lambda Y e_1 e_2 e_1 Y = \lambda Y e_1 Y = i(e_0)$

□

By composition of i with the Markov trace $\text{tr} : \text{BB}'_n \rightarrow R$ we get a trace on BD_n , i.e. we have the following relations:

$$\text{tr}(1) = 1 \quad (35)$$

$$\text{tr}(a e_{n-1}) = x^{-1} \text{tr}(a), a \in \text{BD}_{n-1} \quad (36)$$

$$\text{tr}(a X_{n-1}^\pm) = x^{-1} \lambda^\mp \text{tr}(a) \quad (37)$$

The classical limit ($q = 1 = \lambda, \delta = 0$, x becoming independent of λ) has a graphical interpretation as Brauer diagrams with dots on it. Each arc cannot have more than one dot (due the quadratic relation of Y). The anti-involution considered above amounts to a top-down reflection. The image of BD_n consists of dotted Brauer diagrams with an even number of dots on it. Vertical arcs in a generate vertical arcs and un-dotted arcs in aa^* . Horizontal arcs in a on the side joined together produce un-dotted circles in aa^*st . In the classical limit the trace is given by closing the strings from the right.

Now we can state an important lemma:

Lemma 7 *The trace is non-degenerate on $\text{BD}_n = \text{BP}_n^k$.*

Proof: Let $\{v_i \mid i = 1, \dots, k^{n-1}(2n-1)!!\}$ be a linear basis of dotted Brauer graphs. It suffices to show $\det(\text{tr}(v_i v_j^*)_{i,j}) \neq 0$.

The closure of aa^* is free of dots. Now assume that a has s upper (and hence s lower) horizontal arcs. Then there are s cycles in aa^* . Upon closing, another s cycles are produced from the remaining horizontal arcs. The vertical arcs of a form a permutation and a^* contains the inverse permutation. Upon closing, these $n - 2s$ vertical arcs yield $n - 2s$ cycles. The closure of aa^* has therefore a total of n cycles and $\text{tr}(aa^*) = 1$. For all other dotted Brauer diagrams $b \neq a^*$ the closure of ab will have less than n cycles and hence trace $\text{tr}(ab)$ is a Laurent polynomial in x that with lower degree in x . Hence, there is a unique term with highest power in x and its coefficient can't cancel. Thus the determinant will contain this one term and is therefore not 0. If it is not 0 in the classical limit, it cannot be zero in the general case as well. \square

Next, we determine the structure of BD_n in the generic case. It will turn out to be semi-simple and of dimension $2^{n-1}(2n-1)!!$.

Proposition 8 *Let R be a ring as above and K its field of fractions. The algebra $\text{BD}_n = \text{BD}_n(K)$ is semi-simple and its simple components are indexed by the sets I from proposition 4 for $n, n-2, n-4, \dots$*

The proof uses the same techniques as [5],[3].

Proof: Induction starts from BD_1 . This algebra has basis $\{1, e_0, X_0\}$ and has three 1-d representations given by the three eigenvalues of X_0 . BD_2 has basis $\{1, e_0, e_1, X_0, X_1, X_0 X_1\}$ which is obviously closed under multiplication. X_0 and X_1 each have 3 eigenvalues so that we could expect to have 9 one-dimensional representations. However, the symmetry between X_0, X_1 makes representations depend only on the set of eigenvalues.

Assume the proposition is shown by induction for BD_n .

We apply Jones-Wenzl theory [4],[5] to the following inclusion $\text{BD}_{n-1} \subset \text{BD}_n \subset \text{BD}_{n+1}$. The idempotent is $e = x^{-1}e_n$. This is possible because $\text{BD}_{n-1}, \text{BD}_n$ are semi-simple algebras with a faithful trace by induction assumption. All required properties needed for e have already been established. Jones-Wenzl theory asserts the semi-simplicity of the ideal I_{n+1} generated by e . The quotient algebra BD_{n+1}/I_{n+1} is HD_{n+1} and is semi-simple. Since we work over a field we can conclude that BD_{n+1} is semi-simple and that it is isomorphic to the direct sum $\text{BD}_{n+1} = I_{n+1} \oplus \text{BD}_{n+1}/I_{n+1} = I_{n+1} \oplus \text{HD}_{n+1}$. Jones-Wenzl theory further implies that the simple components of I_{n+1} are indexed by the set of modules of HD_{n-1} . \square

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